New families of conservative systems on S^2 possessing an integral of fourth degree in momenta

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1 Introduction

There is a well-known example of integrable conservative system on S^2, the case of Kovalevskaya [10] in the dynamics of a rigid body, possessing an integral of fourth degree in momenta, see [2]. Using the explicit formular for the total energy of this system

$$H = \frac{du_1^2 + du_2^2 + 2du_3^2}{2u_1^2 + 2u_2^2 + u_3^2} - u_1,$$

where S^2 is given by $u_1^2+u_2^2+u_3^2 = 1$, (see [4]), we can rewrite H in polar coordinates as follows

$$H = \gamma_1(r)(r^2 d\varphi^2 + dr^2) - \gamma_2(r)\cos\varphi \tag{1}$$

where γ_1, γ_2 are some functions and $\gamma_2(r) \neq 0$ for $0 < r < \infty$.

Goryachev proposed in [5] a family of examples of conservative systems on S^2 possessing an integral of fourth degree in momenta. In polar coordinates the total energy can be written as follows

$$H = \rho_0(r, B_1, B_2)(r^2 d\varphi^2 + dr^2) - \rho_1(r)(B_1 \sin 2\varphi + B_2 \cos 2\varphi) - \rho_2(r)\cos\varphi \tag{2}$$

where B_1, B_2 are constants and ρ_0, ρ_1, ρ_2 are some functions. It has been shown in [5] that this family reduced to the case of Kovalevskaya when $B_1 = B_2 = 0$.

We say that a conservative system on S^2 is *smooth* if the Hamiltonian is a sum of a smooth Riemannian metric on S^2 (kinetic energy) and a smooth function U on S^2 (potential energy or simply potential).

In this sence above examples of Kovalevskaya and Goryachev are smooth conservative systems on S^2.

In this paper we proposed new examples of *smooth* conservative systems on S^2 possessing an integral of *fourth* degree in momenta.

In our examples we use the solution ψ_0 of the following initial value problem

$$\psi'''\psi' + 2\psi''^2 - 3\psi^2 = 0, \quad \psi(0) = 0, \psi'(0) = 1, \psi''(0) = 0. \tag{3}$$

We set $\Psi_0(r) = \psi_0(\log r)$, $\Psi_1(r) = \psi_0'(\log r)$, $\Psi_2(r) = \psi_0''(\log r)$.

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We prove the following theorems.

Theorem 1.1 *The Hamiltonian*

$$H = \frac{1}{\Psi_1^2(r)}\left(d\varphi^2 + \frac{dr^2}{r^2}\right) - \Psi_1^2(r)(\Psi_2(r) - \Psi_0(r))\cos\varphi \qquad (4)$$

defines a smooth conservative system on S^2 in polar coordinates. This system possesses an integral of fourth degree and does not possess an integral quadratic or linear in momenta.

Hamiltonian (4) cannot be obtained from (1) and (2) by a change of variables on S^2.

Theorem 1.2 *There is a constant p_0 such that for any $p > p_0$ the Hamiltonians*

$$H_p = \frac{\Psi_1^2(r) - \Psi_0^2(r) + p}{\Psi_1^2(r)}\left(d\varphi^2 + \frac{dr^2}{r^2}\right) - \frac{\Psi_1^2(r)(\Psi_2(r) - \Psi_0(r))\cos\varphi}{\Psi_1^2(r) - \Psi_0^2(r) + p} \qquad (5)$$

define a one-parameter family of smooth conservative system on S^2. These systems possess an integral of fourth degree and does not possess an integral quadratic or linear in momenta.

Hamiltonians (5) cannot be obtained from (1) and (2) by a change of variables on S^2.

In [11] we proposed a new *one-parameter* family of smooth conservative systems on S^2 possessing an integral *cubic* in momenta.

The paper is organized as follows. In the first chapter we consider the initial value problem (3), show the existence of the solution ψ_0 on $\mathbf{R}$ and investigate its asymptotic behaviour at infinity. In the second chapter we obtain a local criterion for the integrability of a geodesic flow with an integral of fourth degree and find a class of metrics with integrable geodesic flows. Finally, in the third chapter we prove Theorem 1.1 and Theorem 1.2.

2 Existence and asymptotic behaviour of the solution

In this chapter we consider initial value problem (3), prove the existence of the solution ψ_0 on $\mathbf{R}$ and investigate its asymptotic behaviour at infinity.

Theorem 2.1 *The solution $\psi_0(y)$ of initial value problem (3) exists for any $y \in \mathbf{R}$ and, moreover, there are C^∞ functions ν and μ such that*

$$\psi_0'(y) = (\exp y)\nu(\exp(-2y)) = \exp(-y)\nu(\exp(2y)),$$

$$\psi_0'^2(y)(\psi_0''(y) - \psi_0(y)) = \exp(-y)\mu(\exp(-2y)) = -(\exp y)\mu(\exp(2y))$$

and $\nu(t) > 0$ for any t.

Proof. Initial value problem (3) has a unique solution $\psi(y) = \psi_0(y)$ which is positive on $(0, \varepsilon)$ and negative on $(-\varepsilon, 0)$ for sufficiently small ε.

Let us consider the case $y > 0$.

Let $R(y) = \ln \psi(y)$ and $q(y) = R'(y)$. Then the differential equation from (3) is equivalent to the following system of differential equations of the first order:

$$\dot{q} = p, \quad \dot{p} = \frac{1}{q}\left(-3q^4 - 7q^2 p - 2p^2 + 3\right). \tag{6}$$

System (6) has two singular points:the knots $p = 0, q = \pm 1$.

Since system (6) is symmetric with respect to $q \mapsto -q$, $y \mapsto -y$, it suffices to consider the case $q > 0$.

Show that orbit Γ of (6), related to $\psi_0(y)$ for $y \geq 0$, converges to the point $q = 1, p = 0$ as $y \to +\infty$.

For $\psi_0(y)$ we have $R(y) \to -\infty$ as $y \to 0+$ and then $q(y) = R'(y) \to +\infty$ as $y \to 0+$.

Write $\Gamma = \{(q,p) | p = -q^2 + qg(q^{-1})\}$ and denote $s = q^{-1}$.

By computation we obtain from (6) the following differential equation for g:

$$g'(1 - gs) + 3g^2 - 3s^2 = 0$$

where the initial condition is the following

$$\lim_{s \to 0+} g(s) = \lim_{q \to +\infty} \frac{p + q^2}{q} = \frac{\psi_0''(0)}{\psi_0'(0)} = 0.$$

So, we have rewritten (3) as the following initial value problem

$$g'(1 - gs) + 3g^2 - 3s^2 = 0, \quad g(0) = 0 \tag{7}$$

The solution of (7) is $g(s) = s^3$. So,

$$\Gamma = \{(q,p) | p = -q^2 + \frac{1}{q^2}\} \tag{8}$$

Since $p(1) = 0$ in (8), orbit Γ converges to the point $q = 1, p = 0$.

Consider the following linear system of differential equations, related to (6).

$$\dot{q} = p, \quad \dot{p} = -12(q - 1) - 7p. \tag{9}$$

The eigenvalues of (9) are equal to -3 and -4. Therefore (6) is C^∞-conjugate to (9) in a neighborhood of $q = 1, p = 0$, see [8]. Taking into account (8), we conclude that there exists C^∞-functions ξ_1, ξ_2 such that the solution of (6), related to $\psi_0(y)$, $y > 0$, can be written as

$$q = 1 + \xi_1(\exp(-4y)),$$
$$p = \xi_2(\exp(-4y))$$

where $\xi_1(0) = \xi_2(0) = 0$.

So, we get

$$\psi_0(y) = (\exp y)\beta(\exp(-2y))$$

where

$$\beta(u) = \exp(-\frac{1}{2}\int^{u} \frac{\xi_1(t^2)}{t} dt).$$

Thus, β is of class C^∞.

Since

$$\psi_0'(y) = \psi_0(y)q = (\exp y)\beta(\exp(-2y))(1 + \xi_1(\exp(-4y)) = (\exp y)\nu(\exp(-2y)),$$

ν is of class C^∞. Since the values of q for orbit Γ belong to $(1, +\infty)$ and $\psi_0'(0) = 1$, the function ψ_0' is positive everywhere. Therefore, $\nu(t) > 0$ for any $t \geq 0$.

Then we compute

$$\psi_0'^2(y)(\psi_0''(y) - \psi_0(y)) = \psi_0'^2(y)\psi_0(y)(p + q^2 - 1) =$$

$$\exp(-y)\nu^2(\exp(-2y))\beta(\exp(-2y))\frac{\xi_2(\exp(-4y)) + 2\xi_1(\exp(-4y)) + \xi_1^2(\exp(-4y))}{\exp(-4y)}$$

$$= \exp(-y)\mu(\exp(-2y))$$

and, therefore, μ is of class C^∞.

Since $\psi_0(y) = -\psi_0(-y)$, we get

$$\psi_0'(y) = (\exp y)\nu(\exp(-2y)) = \exp(-y)\nu(\exp(2y))$$

and

$$\psi_0'^2(y)(\psi_0''(y) - \psi_0(y)) = \exp(-y)\mu(\exp(-2y)) = -(\exp y)\mu(\exp(2y)).$$

$\square$

3 A criterion for the integrability

Consider a metric $ds^2 = \Theta(u, v)(du^2 + dv^2)$ in conformal coordinates u, v. It can also be written as

$$ds^2 = \theta(w, \bar{w})dwd\bar{w} \tag{10}$$

where $w = u+iv$. The geodesic flow of ds^2 is a Hamiltonian system with Hamiltonian

$$H = \frac{p_w p_{\bar{w}}}{4\theta(w, \bar{w})}. \tag{11}$$

A polynomial F in momenta p_u, p_v can be also written as

$$F = \sum_{k=0}^{n} b_k(w, \bar{w}) p_w^k p_{\bar{w}}^{n-k}$$

where

$$b_k = \overline{b_{n-k}}, \quad k = 0, ..., n.$$

If the polynomial F is an additional integral of the geodesic flow with Hamiltonian (11), then $\{F, H\} = 0$ and the following holds

$$\theta \frac{\partial b_{k-1}}{\partial w} + (n-(k-1))b_{k-1}\frac{\partial \theta}{\partial w} + \theta \frac{\partial b_k}{\partial \bar{w}} + kb_k \frac{\partial \theta}{\partial \bar{w}} = 0, \tag{12}$$

where $k = 0, ..., n+1$ and $b_{-1} = b_{n+1} = 0$. Substituting $k = 0$ and $k = n+1$ in (12) we get immediately

$$\frac{\partial}{\partial \bar{w}} b_0 \equiv 0$$

and

$$\frac{\partial}{\partial w} b_{n+1} \equiv 0.$$

Note that in another conform coordinate system

$$\hat{a}_0(\hat{w}) = a_0(w)\left(\frac{d\hat{w}}{dw}\right)^n. \tag{13}$$

Proposition 3.1 *The geodesic flow of a Riemannian metric on S^2*

$$ds^2 = \lambda(r^2)(r^2 d\varphi^2 + dr^2) \tag{14}$$

does not possess a nontrivial integral quadratic in momenta (which does not depend on H and linear integrals).

Proof. Let us introduce the conform coordinate system $u = r\cos\varphi, v = r\sin\varphi$. Denote $w = u + iv$ and consider $\hat{w} = w^{-1}$ for ds^2.

Let $F = a_0(w,\bar{w})p_w^2 + a_1(w,\bar{w})p_w p_{\bar{w}} + a_2(w,\bar{w})p_{\bar{w}}^2$ be an integral of the geodesic of (14). As above, $a_0 = a_0(w)$, $a_2 = a_2(\bar{w}) = \bar{a}_0(w)$ and Im $a_1(w,\bar{w}) = 0$. Since (14) is a Riemannian metric on S^2, from (13) it follows that $a_0(w)$ is a polynomial of degree n where $n \le 4$. If F does not depend on the Hamiltonian and linear integrals, then $a_0(w)$ does not equal to zero identically.

Then from (12) when $k = 1$ we obtain

$$\lambda \frac{\partial a_0}{\partial w} + 2a_0 \frac{\partial \lambda}{\partial w} + \lambda \frac{\partial a_1}{\partial \bar{w}} + a_1 \frac{\partial \lambda}{\partial \bar{w}} = 0.$$

This is equivalent to the following condition

$$\text{Im } \frac{\partial}{\partial w}\left(\lambda \frac{\partial a_0}{\partial w} + 2a_0 \frac{\partial \lambda}{\partial w}\right) = 0$$

where $a_0(w) = A_0 w^4 + A_1 w^3 + A_2 w^2 + A_3 w + A_4$ and A_0, A_1, A_2, A_3, A_4 are some constants such that at least one of them does not equal to zero.

Denote $w\bar{w}$ as t. So, we get the following conditions for $\lambda(t)$

$$\text{Im } (w^2(12A_0\lambda + 12A_0\lambda' t + 2A_0\lambda'' t^2) + \bar{w}^2(2A_4\lambda''))+$$

$$\text{Im }(w(6A_1\lambda + 9A_1\lambda' t + 2A_1\lambda'' t^2) + \bar{w}(3A_3\lambda' + 2A_3\lambda'' t)) +$$
$$\text{Im }(A_2(2\lambda + 6\lambda' t + 2\lambda'' t^2)) \equiv 0.$$

So, we have to consider the following linear differential equations

$$\epsilon_1(6\lambda + 6\lambda' t + \lambda'' t^2) = \epsilon_2(\lambda''), \tag{15}$$

$$\epsilon_1(6\lambda + 9\lambda' t + 2\lambda'' t^2) = \epsilon_2(3\lambda' + \lambda'' t), \tag{16}$$

$$\lambda + 3\lambda' t + 2\lambda'' t^2 = 0 \tag{17}$$

where ϵ_1, ϵ_2 are constants equal to -1, 0 or 1. In another case consider the rescaling $t \mapsto Dt$, D is a real constant, of a solution (15) or (16)).

It is easy to see that for any smooth solution of (17) $\lambda(0) = 0$ and then ds^2 is not a Riemannian metric on S^2. (We get that Im $A_2 = 0$.)

It is also easy to see that any solution of the equations (15), (16) if $\epsilon_1 = 0$ or $\epsilon_2 = 0$ does not define by (14) a Riemannian metric on S^2.

Solutions of (15) if $\epsilon_1 = 1$ and $\epsilon_2 = 1$ have the following form

$$\lambda(t) = C_1 \frac{1+t^2}{(-1+t^2)^2} + C_2 \frac{t}{(-1+t^2)^2}, C_1, C_2 - const.$$

Solutions of (15) if $\epsilon_1 = 1$ and $\epsilon_2 = -1$ have the following form

$$\lambda(t) = C_1 \frac{1-t^2}{(1+t^2)^2} + C_2 \frac{t}{(1+t^2)^2}, C_1, C_2 - const.$$

We got it noting that solutions of (15) if $\epsilon_1 = 1$ and $\epsilon_2 = 1$ can be obtained from the solutions of (15) if $\epsilon_1 = 1$ and $\epsilon_2 = -1$ by complex transformation $t \to it$.

It is easy to note that if $\lambda(0) = 1$ there is only one smooth solution of (16) if $\epsilon_1 = 1$ or $\epsilon_2 = \pm 1$. These solutions correspond to the metrics of curvature -1 if $\epsilon_2 = 1$ and to the metrics of curvature -1 if $\epsilon_2 = -1$. So, if $\epsilon_1 = 1$ and $\epsilon_2 = 1$ any solution of (16) which is smooth in zero has the following form

$$\lambda(t) = \frac{C_1}{(-1+t)^2}, C_1 - const$$

and if $\epsilon_1 = 1$ and $\epsilon_2 = -1$ any solution of (16) which is smooth in zero has the following form

$$\lambda(t) = \frac{C_1}{(1+t)^2}, C_1 - const.$$

Thus, we may see that either Im $A_2 = 0$ and $A_i = 0, i \neq 2$ or

$$\lambda(t) = \frac{C_1}{(-1+Dt)^2}, C_1, D - const$$

and, therefore,

$$ds^2 = \frac{C_1}{(1+Dr^2)^2}\left(r^2 d\varphi^2 + dr^2\right), \tag{18}$$

where $C_1, D - const$, i.e. ds^2 is a metric of constant positive curvature.

Remember that the geodesic flow of a metric (14) possesses the linear integral $F_1 = iwp_w - i\bar{w}p_{\bar{w}}$. If Im $A_2 = 0$ and $A_i = 0, i \neq 2$, we will consider the integral $F_2 = F + A_2 F_1$ which is equal, obviously, $\hat{C}H$ where $\hat{C}$ is a constant. Therefore, F depends on H and F_1.

So, if the geodesic flow of a metric (14) on S^2 possesses a nontrivial quadratic integral, it can be only a metric of constant positive curvature. But it is well known that the geodesic flows of metrics of constant curvature possess two independent linear integrals. (Indeed, the geodesic flows with the Hamiltonians $H = l'^2(y)(p_x^2 + p_y^2)$ where $l''(y) = l(y)$ possess the integrals $F_1 = p_x$ and $\hat{F}_1 = l(y)\cos xp_x - l'(y)\sin xp_y$.) So, in this case any quadratic integral is also trivial, i.e. depends on the Hamiltonian and linear integrals. □

Proposition 3.2 *The geodesic flow of a metric (14) on S^2 possesses two independent linear integrals if and only if it has form (18), i.e. it is a metric of constant positive curvature.*

Proof. As mentioned above the geodesic flow of a metric (14) on S^2 possesses the linear integral $F_1 = p_\varphi$.

In the coordinates $w, \bar{w}, p_w, p_{\bar{w}}$ where $w = r\cos\varphi + ir\sin\varphi$ it can be written as follows $F_1 = iwp_w - i\bar{w}p_{\bar{w}}$. (Indeed, in the coordinates $z = \varphi + iy$, $y = \log r$ we have $F_1 = p_z + p_{\bar{z}}$ and, clearly, $w = \exp iz$.)

Assume that a metric ds^2 of form (14) on S^2 possesses another linear integral $F_2 = b_0(w)p_w + \bar{b}_0(w)p_{\bar{w}}$ which is independent of F_1 and, therefore, the function $b_0(w)$ does not equal to iwC_3 where C_3 is a real constant.

Let us consider then the function $F_2^2 = b_0^2(w)P_w + 2|b_0|^2 p_w p_{\bar{w}} + \bar{b}_0(w)p_{\bar{w}}^2$ which is also an integral of the geodesic flow of ds^2.

Compare now $b_0^2(w)$ with $a_0(w)$ from Proposition 3.1. Since $b_0(w) \neq iwC_3$ in our case, then at least one coefficient A_i, $i \neq 2$ (see Proposition 3.1) does not equal to zero. In the same way as in Proposition 3.1 we may show that ds^2 has form (18), i.e. ds^2 is a metric of constant positive curvature.

□

Remember that conform coordinates x, y of a metric ds^2 are Liouville if $ds^2 = (f(x) + h(y))(dx^2 + dy^2)$ for some functions f, h. Due to Darboux [6] Liouville coordinates exist if and only if the geodesic flow of ds^2 possesses an integral quadratic in momenta and moreover if one of the function f or h is constant then the geodesic flow possesses an integral linear in momenta. So, we may formulate the following

Corollary 3.3 *Liouville coordinates $\varphi, y = \log r$, related to polar coordinates φ, r of a metric (14) on S^2 are unique up to shifts and the transform $y \to -y$.*

Proof. Assume that a metric ds^2 on S^2 can be written in two different forms (14)

$$ds^2 = \lambda_1(r^2)(r^2 d\varphi^2 + dr^2)$$

and

$$ds^2 = \lambda_2(\tilde{r}^2)(\tilde{r}^2 d\tilde{\varphi}^2 + d\tilde{r}^2)$$

where $\tilde{r} \neq Dr^{\pm 1}$, $D - const$. This means that the geodesic flow of ds^2 possesses two independent linear integrals. So, from Proposition 3.2 it follows that $\tilde{r} \neq Dr^{\pm 1}$, $D-$ *const* and, therefore, Liouville coordinates $\varphi, y = \log r$ related to polar coordinates φ, r of ds^2 on S^2 are unique up to shifts and the transform $y \to -y$.

□

We must note that Proposition 3.1 follows from the results of Kolokol'tsov published in his Ph.D. thesis (in Russian) but we give here a complete proof.

One may show that if the polynomial in momenta integral F of the geodesic flow of (10) is independent of the Hamiltonian and an integral of smaller degree, then there is a conformal coordinate system $z = z(w)$ of this metric such that the coefficients of F for p_z and $p_{\bar{z}}$ equal 1 identically.

Theorem 3.4 *Let $ds^2 = \lambda(z,\bar{z})dzd\bar{z}$ be a metric such that there exists a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$, satisfying the following conditions*

$$\lambda(z,\bar{z}) = \frac{\partial^2 f}{\partial z \partial \bar{z}}, \quad Im \left(\frac{\partial^4 f}{\partial z^4}\frac{\partial^2 f}{\partial z \partial \bar{z}} + 3\frac{\partial^3 f}{\partial z^3}\frac{\partial^3 f}{\partial z^2 \partial \bar{z}} + 2\frac{\partial^2 f}{\partial z^2}\frac{\partial^4 f}{\partial z^3 \partial \bar{z}} \right) = 0. \tag{19}$$

Then the geodesic flow of ds^2 possesses an integral of fourth degree in momenta.

If the geodesic flow of a metric ds^2 possesses an integral which is a polynomial of fourth degree in momenta and it does not depend on the Hamiltonian and an integral of smaller degree then there exist conformal coordinates x, y and a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ such that $ds^2 = \lambda(z,\bar{z})dzd\bar{z}$ where $z = x + iy$ and (19) holds.

Proof. Suppose that (19) holds. We will now construct an integral

$$F = \sum_{k=0}^{n} a_k(z,\bar{z}) p_z^k p_{\bar{z}}^{n-k}$$

of the corresponding geodesic flow.

Put $a_0 = 1$ and $a_n = 1$. Equation (12) then has the following form

$$4\frac{\partial \lambda}{\partial z} = -\frac{\partial (a_1 \lambda)}{\partial \bar{z}}, \tag{20}$$

$$\frac{\partial (a_1 \lambda^3)}{\partial z} = -\lambda \frac{\partial (a_2 \lambda^2)}{\partial \bar{z}}, \tag{21}$$

$$a_2 = \overline{a_2}. \tag{22}$$

Consider

$$a_1 = -4\frac{\partial^2 f}{\partial z^2}\left(\frac{\partial^2 f}{\partial z \partial \bar{z}}\right)^{-1}.$$

Then (20) holds.

Show that for

$$g = \frac{\partial (a_1 \lambda^3)}{\partial z}\lambda^{-1}$$

it holds

$$\mathrm{Im}\ \frac{\partial g}{\partial z} = 0.$$

Indeed, by computation we obtain

$$\mathrm{Im}\ \frac{\partial g}{\partial z} = -4\ \mathrm{Im}\left(\frac{\partial^4 f}{\partial z^4}\frac{\partial^2 f}{\partial z \partial \bar{z}} + 3\frac{\partial^3 f}{\partial z^3}\frac{\partial^3 f}{\partial z^2 \partial \bar{z}} + 2\frac{\partial^2 f}{\partial z^2}\frac{\partial^4 f}{\partial z^3 \partial \bar{z}}\right) = 0.$$

So, it follows that there is a function $h : \mathbf{R}^2 \mapsto \mathbf{R}$ such that

$$\frac{\partial h}{\partial \bar{z}} = g.$$

Consider $a_2 = -\frac{h}{\lambda^2}$. Then $a_2 : \mathbf{R}^2 \mapsto \mathbf{R}$ and, moreover,

$$\frac{\partial(a_1 \lambda^3)}{\partial z}\lambda^{-1} = -\frac{\partial(a_2 \lambda^2)}{\partial \bar{z}},$$

i.e. conditions (21) and (22) hold also.

Put then $a_3 = \bar{a}_1$. So, we have an integral of fourth degree in momenta of the geodesic flow of ds^2.

Prove now the second statement of the theorem. As mentioned above, in this case there exists a conformal coordinate system of ds^2 where $a_0 = a_4 \equiv 1$. Note that this system is unique up to a shift. So, in this coordinate system (20)-(22) hold. From (20) it follows that there exists a function V where

$$\frac{\partial V}{\partial \bar{z}} = \lambda$$

and

$$a_1 \lambda = -4\frac{\partial V}{\partial z}.$$

From (21) and (22) it follows immediately that

$$\mathrm{Im}\ \frac{\partial}{\partial z}\left(\frac{\partial(a_1 \lambda^3)}{\partial z}\lambda^{-1}\right) = 0.$$

On the other hand

$$\mathrm{Im}\ \frac{\partial}{\partial z}\left(\frac{\partial(a_1 \lambda^3)}{\partial z}\lambda^{-1}\right) = -4\ \mathrm{Im}\ \frac{\partial}{\partial z}\left(\frac{\partial^2 V}{\partial z^2}\frac{\partial V}{\partial \bar{z}} + 2\frac{\partial^2 V}{\partial z \partial \bar{z}}\frac{\partial V}{\partial z}\right) = 0.$$

Taking into account that $\mathrm{Im}\ \frac{\partial V}{\partial \bar{z}} = \mathrm{Im}\ \lambda = 0$, we conclude that there exists a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ such that

$$\frac{\partial f}{\partial z} = V.$$

Thus,

$$\lambda = \frac{\partial^2 f}{\partial z \partial \bar{z}}$$

and
$$0 = \mathrm{Im}\ \frac{\partial}{\partial z}\left(\frac{\partial^3 f}{\partial z^3}\frac{\partial^2 f}{\partial z\partial\bar z} + 2\frac{\partial^3 f}{\partial z^2\partial\bar z}\frac{\partial^2 f}{\partial z^2}\right) =$$
$$= \mathrm{Im}\ \left(\frac{\partial^4 f}{\partial z^4}\frac{\partial^2 f}{\partial z\partial\bar z} + 3\frac{\partial^3 f}{\partial z^3}\frac{\partial^3 f}{\partial z^2\partial\bar z} + 2\frac{\partial^2 f}{\partial z^2}\frac{\partial^4 f}{\partial z^3\partial\bar z}\right).$$
□

Equation (19) in some other form has been written in [7] but in order to show when the corresponding integral is nontrivial we needed to give here a complete proof.

Using this criterion we may find families of metrics with integrable geodesic flows.

We will look for a class of solutions of (19) and show that in this class of solutions the problem of integrability can be reduced to an ordinary differential equation.

Denote
$$\frac{\partial^2 f}{\partial z^2} = (A(x,y) + iB(x,y))\,.$$
By computation we obtain
$$\mathrm{Im}\ \frac{\partial^4 f}{\partial z^4} = \frac{1}{4}\left(B_{xx} - B_{yy} - 2A_{xy}\right),$$
$$\mathrm{Im}\ \frac{\partial^3 f}{\partial z^3}\frac{\partial^3 f}{\partial z^2\partial\bar z} = \frac{1}{2}\left(A_x B_x + A_y B_y\right),$$
$$\mathrm{Im}\ \frac{\partial^2 f}{\partial z^2}\frac{\partial^4 f}{\partial z^3\partial\bar z} = \frac{1}{4}\left(B(A_{xx} + A_{yy}) + A(B_{xx} + B_{yy})\right).$$
Thus, (19) can be written as follows
$$6(A_yB_y + A_xB_x) + 2(B(A_{xx} + A_{yy}) + A(B_{xx} + B_{yy})) + (B_{xx} - B_{yy} - 2A_{xy})\lambda = 0. \quad (23)$$
Consider now the solutions of (19) of the following form:
$$f(x,y) = \psi(y)\cos x + \xi(y) + d(x^2 - y^2) \quad (24)$$
where ψ, ξ are some smooth functions and d is a constant.

Then we obtain
$$\lambda = \frac{1}{4}\left((\psi''(y) - \psi(y))\cos x + \xi''(y)\right)$$
and
$$A = -\frac{1}{4}\left((\psi''(y) + \psi(y))\cos x + \xi''(y) - 4d\right), B = \frac{1}{2}\left(\psi'(y)\sin x\right). \quad (25)$$
Substituting (25) in (23) we get the following condition of integrability
$$6(((\psi''' + \psi')\cos x + \xi''')2\psi''\sin x - (\psi'' + \psi)2\psi'\sin x\cos x +$$
$$2(2\psi'\sin x((\psi^{(4)} - \psi)\cos x + \xi^{(4)}) + ((\psi'' + \psi)\cos x + \xi'' - 4d)2(\psi''' - \psi')\sin x)) +$$

$$4(\psi''' + \psi')\sin x((\psi'' - \psi)\cos x + \xi'') \equiv 0.$$

Therefore,

$$3\psi''\xi''' + \psi'\xi^{(4)} + 2\psi'''\xi'' = 4d(\psi''' - \psi'), \tag{26}$$

$$5\psi'''\psi'' - 6\psi'\psi + \psi'\psi^{(4)} = 0. \tag{27}$$

Equation (26) can be written as

$$(\psi'\xi''')' + 2(\psi''\xi'')' = 4d(\psi'' - \psi)'.$$

By integrating we get

$$\psi'\xi''' + 2\psi''\xi'' = 4d(\psi'' - \psi) + d_1$$

where d_1 is a constant. So, if $d = 0$

$$\xi'' = \frac{d_1\psi(y) + c}{(\psi'(y))^2}. \tag{28}$$

or if $d \neq 0$

$$\xi'' = 2d\frac{\psi'^2(y) - \psi^2(y) + d_1(2d)^{-1}\psi(y) + p}{\psi'(y))^2}. \tag{29}$$

where p, c are some constants.

On the other hand there is an integral of (27):

$$2\psi''^2 - 3\psi^2 + \psi'\psi''' = const. \tag{30}$$

So, we may now formulate the following proposition.

Proposition 3.5 *For any solution ψ of (30) there are two families of metrics $ds_1^2 = \Lambda(x, y)(dx^2 + dy^2)$ where*

$$\Lambda(x, y) = (\psi''(y) - \psi(y))\cos x + \frac{d_1\psi(y) + c}{\psi'^2(y)} \tag{31}$$

or

$$\Lambda(x, y) = (\psi''(y) - \psi(y))\cos x + c\frac{\psi'^2(y) - \psi^2(y) + d_1\psi(y) + p}{\psi'^2(y)} \tag{32}$$

where c, d, p are constants and the corresponding geodesic flows possess an integral of fourth degree in momenta. If $\frac{\partial^2\Lambda}{\partial x\partial y}$ and $\frac{\partial^2\Lambda}{\partial x^2} - \frac{\partial^2\Lambda}{\partial y^2}$ are not equal zero identically, then this integral is nontrivial, i.e. it does not depend on the Hamiltonian and an integral of smaller degree.

Proof. As we have shown, for every metric in one of these families there exists a function f of the form (24) satisfying (19). In fact, compare (31) with (28) and (32) with (29) (in the last case we denote $4d$ in (29) as c).

Since every metric in one of these families satisfies the conditions of theorem 3.4, the integrability follows.

So, we have only to prove that the integrals of the geodesic flows of these geodesic flows are nontrivial.

According to our construction the integral F of the geodesic flow with Hamiltonian

$$H = \frac{p_z p_{\bar z}}{4\Lambda}$$

where Λ is given by (31), (32) has the following form

$$F = p_z^4 + D(x,y)p_z^3 p_{\bar z} + E(x,y)p_z^2 p_{\bar z}^2 + \bar D(x,y)p_z p_{\bar z}^3 + p_{\bar z}^4$$

where $D(x,y) : \mathbf{R^2} \mapsto \mathbf{C}$, $E(x,y) : \mathbf{R^2} \mapsto \mathbf{R}$.

Thus, if F depends on H and another integral polynomial of smaller degree, then there is an integral $\tilde F$ quadratic in momenta which has the following form

$$\tilde F = p_z^2 + \tilde E_1(x,y)p_z^2 p_{\bar z}^2 + p_{\bar z}^2$$

or

$$\tilde F = ip_z^2 + \tilde E_2(x,y)p_z^2 p_{\bar z}^2 - ip_{\bar z}^2$$

where $\tilde E_1(x,y), \tilde E_2(x,y) : \mathbf{R^2} \mapsto \mathbf{R}$.

Therefore, due to Darboux [6] the following holds

$$\frac{\partial^2 \Lambda}{\partial x \partial y} \equiv 0$$

or

$$\frac{\partial^2 \Lambda}{\partial x^2} - \frac{\partial^2 \Lambda}{\partial y^2} \equiv 0.$$

$\square$

4 Smooth metrics on S^2

In this chapter we will prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Prove that Hamiltonian system with Hamiltonian (4) is a smooth system on S^2. Using theorem 2.1, we may write

$$H = \frac{1}{\nu^2(r^2)}(r^2 d\varphi^2 + dr^2) + \mu(r^2) r\cos\varphi =$$

$$\frac{1}{\nu^2(\tilde r^2)}(\tilde r^2 d\tilde\varphi^2 + d\tilde r^2) - \mu(\tilde r^2)\tilde r\cos\tilde\varphi$$

where $\tilde r = \frac{1}{r}$, $\tilde\varphi = -\varphi$. Since ν, μ are of class C^∞ and $\nu \neq 0$, see Theorem 2.1, then this system is a smooth system on S^2.

Prove that (4) possesses an integral of fourth degree in momenta. Consider metrics (31) from Proposition 3.5 and put in (31) $\psi(y) = \psi_0(y)$ and $d_1 = 0$. Then we obtain the following one-parameter family of metrics:

$$\left((\psi_0''(y) - \psi_0(y))\cos x + c\frac{1}{\psi_0'^2(y)}\right)(dx^2 + dy^2)$$

or in polar coordinates $\varphi = x$, $r = \exp y$:

$$\left((\Psi_2(r) - \Psi_0(r))\cos\varphi + c\frac{1}{\Psi_1^2(r)}\right)(d\varphi^2 + \frac{1}{r^2}dr^2)$$

where c is an arbitrary constant. From Proposition 3.5 we know that any metric in this family possesses an integral of fourth degree in momenta. Using the well-known Maupertuis's principle, see [1], we conclude that (4) possesses an integral of fourth degree in momenta, see also [3], [4].

So, we have to prove only that this integral nontrivial, i.e. depends on the Hamiltonian and an integral of smaller degree. Denote the kinetik energy in (4) as $K = \lambda(y)(d\varphi^2 + dy^2)$ and the potential as V. We will denote the geodesic flow of the metric of K as K also.

Let us assume that a system of this family has an integral which is independent of the Hamiltonian (total energy (4)) and which is a polynomial of second degree in momenta (clearly, this assumption includes the case of linear integrals). So, there is an integral $\tilde{F}$ of (4) which is quadratic in momenta. Thus, $\tilde{F} = A(p_\varphi, p_y, \varphi, y) + B(\varphi, y)$ where $A(p_\varphi, p_y, \varphi, y)$ is a polynomial of second degree in momenta. We may write $\{\tilde{F}, H\} = \{A(p_\varphi, p_y, \varphi, y) + B(\varphi, y), K + V\} \equiv 0$ and, therefore, $\{A(p_\varphi, p_y, \varphi, y), K\} \equiv 0$.

Thus, the geodesic flow of K has an integral which is a polynomial of second degree with respect to momenta.

Since the function ψ such that $\psi'(y) = c_1 \exp(-y)(1 + \exp(2y)), c_1 - const$ does not satisfy (3) and, therefore, $\Psi_1(r) \neq r^{-1}(1 + Dr^2)$, D is a positive constant, then K is not a metric of constant positive curvature. So, the quadratic integral A of the geodesic flow of K depends on the Hamiltonian K and the integral p_φ. W.l.o.g. we may put $A = p_\varphi^2$.

So, we may write

$$\{A, V\} + \{B, K\} = \{p_\varphi^2, V\} + \{B, K\} \equiv 0.$$

By computation we obtain

$$\frac{\partial B}{\partial y} \equiv 0,$$

and

$$\frac{\partial B}{\partial \varphi} = \frac{\partial V}{\partial \varphi}\lambda(y).$$

So, we get

$$V = \frac{B(\varphi) + \alpha(y)}{\lambda(y)}$$

for a smooth function $\alpha(y)$.

In our case

$$\lambda(y) = \frac{1}{\psi_0'^2(y)}$$

and

$$V(x, y) = (\psi_0''(y) - \psi_0(y))\psi_0'^2(y)\cos\varphi$$

and we get then $\psi_0''(y) - \psi_0(y) \equiv const$, that is not true. So, there is no quadratic integral of the system given by (4).

In oder to prove that the system given by (4) is really a new example of the integrability by a polynomial of fourth degree in momenta we have to compare this case with the known cases of Kovalevskaya and Goryachev. First of all, we note that in the family (2) we have to consider only the case $B_1 = B_2 = 0$. As mentioned above, for $B_1 = B_2 = 0$ the system given by (2) is simply the case of Kovalevskaya. So, we may consider only the case of Kovalevskaya. Then comparing the potentials in (1) with the potentials in (4) and (5) we see that they are cannot be the same, because $\gamma_2(r) > 0$, $r \in (0, +\infty)$ in (1) but $\Psi_2(1) - \Psi_0(1) = 0$.

So, this example is really new, i.e. it cannot be obtained from the known integrable coservative systems on S^2 corresponding to the cases of Kovalevskaya and Goryachev.

□

Proof of Theorem 1.2. Consider Hamiltonian (5) and prove that there is a constant p_0 that for any $p > p_0$ it is a smooth system on S^2. Using theorem 2.1, we may rewrite H_p in (5) as

$$H_p = \frac{\Phi(r^2) + p + 1}{\nu^2(r^2)}(r^2 d\varphi^2 + dr^2) + \frac{\mu(r^2)}{\Phi(r^2) + p + 1} r \cos\varphi =$$

$$\frac{-\Phi(\tilde{r}^2) + p + 1}{\nu^2(\tilde{r}^2)}(\tilde{r}^2 d\tilde{\varphi}^2 + d\tilde{r}^2) - \frac{\mu(\tilde{r}^2)}{-\Phi(\tilde{r}^2) + p + 1} \tilde{r} \cos\tilde{\varphi}$$

where $\tilde{r} = \frac{1}{r}$, $\tilde{\varphi} = -\varphi$ and

$$\Phi(t) = \int_t^1 \mu(s)\nu^{-1}(s)ds.$$

Since ν, μ are of class C^∞ and $\nu \neq 0$, see Theorem 2.1, there are constants

$$M_1 = \max_{[0,1]} \Phi(t), \;\; M_2 = -\min_{[0,1]} \Phi(t)$$

and, therefore, for $p > p_0 = \max\{M_1, M_2\} - 1$ system with Hamiltonian (5) is a smooth conservative system on S^2.

Prove that the system with energy (5) possesses an integral of fourth degree in momenta. Consider metrics (32) from Proposition 3.5 and put in (32) $\psi(y) = \psi_0(y)$ and $d_1 = 0$. We get the following two-parameter family of metrics:

$$\left((\psi_0''(y) - \psi_0(y)) \cos x + c \frac{\psi_0'^2(y) - \psi_0^2(y) + p}{\psi_0'^2(y)} \right) (dx^2 + dy^2)$$

which can be written also in polar coordinates $\varphi = x$, $r = \exp y$ as

$$\left((\Psi_2(r) - \Psi_0(r)) \cos\varphi + c \frac{\Psi_1^2(r) - \Psi_0^2(r) + p}{\Psi_1^2(r)} \right) (d\varphi^2 + \frac{1}{r^2} dr^2)$$

where c is an arbitrary constant. It was shown in Proposition 3.5 that any metric in this family possesses an integral of fourth degree in momenta. From the well-known Maupertuis's principle, see [1], it follows that the system with Hamiltonian (5) possesses an integral of fourth degree in momenta.

In order to prove that for any $p > p_0$ the system with Hamiltonian (5) does not possess an integral quadratic or linear in momenta and to compare with the above cases of Kovalevskaya and Goryachev we can apply the same arguments as for the system with Hamiltonian (4), see the proof of Theorem 1.1.

We must prove that for different values of parameter p the systems given by (5) cannot be obtained one from another by a change of variables on S^2, i.e. by a diffeomorphismus $\phi : \ S^2 \to S^2$.

Let us introduce the coordinates $\varphi, y = \log r$. From Corollary 3.3 it follows that we need to consider only the transformations $y \mapsto \pm y + \kappa$ where κ is a constant.

Let us rewrite H_p in the coordinates φ, y:

$$H_p = \frac{\psi_0'^2(y) - \psi_0^2(y) + p}{\psi_0'^2(y)} \left(d\varphi^2 + dy^2\right) -$$

$$(\psi_0''(y) - \psi_0(y)) \left(\frac{\psi_0'^2(y) - \psi_0^2(y) + p}{\psi_0'^2(y)}\right)^{-1} \cos\varphi \tag{33}$$

If the systems given by (33) for $p = p_1$ and $p = p_2$ can be obtained one from another by a transform $y \mapsto \pm y + \kappa$, $\kappa - const$, then there are some constants $K_1 \neq 0$, $K_2 \neq 0$ such that the following holds

$$\frac{\psi_0'^2(y) - \psi_0^2(y) + p}{\psi_0'^2(y)} =$$

$$K_1 \frac{\psi_0'^2(\pm y + \kappa) - \psi_0^2(\pm y + \kappa) + p}{\psi_0'^2(\pm y + \kappa)} \tag{34}$$

and

$$(\psi_0''(y) - \psi_0(y)) \left(\frac{\psi_0'^2(y) - \psi_0^2(y) + p}{\psi_0'^2(y)}\right)^{-1} =$$

$$K_2(\psi_0''(\pm y + \kappa) - \psi_0(\pm y + \kappa)) \left(\frac{\psi_0'^2(\pm y + \kappa) - \psi_0^2(\pm y + \kappa) + p}{\psi_0'^2(\pm y + \kappa)}\right)^{-1}. \tag{35}$$

Thus, we get

$$\psi_0''(y) - \psi_0(y) = K_1 K_2(\psi_0''(\pm y + \kappa) - \psi_0(\pm y + \kappa)). \tag{36}$$

Put $y = 0$ in (36) and we get $\psi_0''(\kappa) - \psi_0(\kappa) = 0$. W.l.o.g. we assume that $\kappa \geq 0$. From the proof of Theorem 2.1 we know that the function $\psi_0''(y) - \psi_0(y)$, $y > 0$ can be written as follows

$$\psi_0''(y) - \psi_0(y) = (\exp R(y))(p(q) + q^2 - q), y > 0$$

where $p(q) = -q^2 + \frac{1}{q^2}$ and $q > 1$.

So, we obtain $\psi_0''(y) - \psi_0(y) = \exp R(y)(q^{-2} - q)$ and, therefore, if $\psi_0''(\kappa) - \psi_0(\kappa) = 0$ and $\kappa \neq 0$, then $q(\kappa) = 1$ that is impossiblle. Thus, $\kappa = 0$. Then taking into account that $\psi_0(y) = -\psi_0(-y)$ from (34) we get $K_1 = 1$ and $p_1 = p_2$ and then from (35) we get $K_2 = 1$.

Thus, for $p_1 \neq p_2$ the Hamiltonians H_p (5) cannot be obtained one from another by a diffeomorphismus $\phi : \ S^2 \to S^2$.

□

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After the paper was finished, K. P. Hadeler informed the author that the differential equation in (3) can be solved by succsessful integration and the inverse function of the solution $\psi_0(y)$ to the initial value problem (3) can be given as:

$$y = y(\psi_0) = \frac{1}{4}\log\frac{(\psi_0^4+1)^{\frac{1}{4}}+\psi_0}{(\psi_0^4+1)^{\frac{1}{4}}-\psi_0} - \frac{1}{2}\arctan\frac{(\psi_0^4+1)^{\frac{1}{4}}}{\psi_0}.$$

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